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On renormalisation of $\lambda\phi^4$ field theory in curved space–time: II

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Abstract. An explicit renormalisation of all second-order physical processes occurring in $\lambda\phi^4$ field theory in conformally flat space–time, including vacuum-to-vacuum processes, is performed. Although divergences dependent on the definition of the vacuum state appear in some Feynman diagrams, physical amplitudes obtained by summing all diagrams which contribute to a single physical process are independent of these divergences. Consequently, the theory remains renormalisable in curved space–time, at least to second order in λ. Renormalisations of the mass m, the coupling constant λ and the constant ξ which couples the field to the Ricci scalar are required to make two- and four-particle creation amplitudes finite. Vacuum-to-vacuum processes are made finite by renormalising coupling constants in a modified Einstein action which includes terms which are quadratic in the curvature. Dimensional regularisation is used to obtain expressions for the formally divergent renormalisation constants for m, ξ and λ. Those for m and λ are the same as in Minkowski space. In order to calculate the divergences in $G^3(x, x')$, a new momentum-space representation, valid in conformally flat space–times, is developed for the Feynman propagator. This is a generalisation to curved space–time of the usual Minkowski space momentum representation.

1. Introduction

In the preceding paper (Bunch *et al* 1980, hereafter referred to as I), we studied the quantum theory of a scalar field in curved space–time with quartic self-interaction, $\lambda\phi^4$. The main purpose of that paper was to generalise normal ordering of field operators to curved space–time and to show that this makes all S-matrix elements and expectation values of the stress tensor finite to first order in λ. We also gave a preliminary discussion of some of the problems that arise in the renormalisation of second-order S-matrix elements, notably the appearance of divergences involving a quantity which depends on the definition of the vacuum state, and we indicated that nevertheless the theory remains renormalisable in curved space–time to second order in λ. The purpose of this paper is to verify this explicitly in an arbitrary conformally flat space–time.

We have divided the discussion of this paper into two parts. The first part (§ 2) is concerned with the renormalisation of Feynman diagrams having two or four external lines, i.e. diagrams which represent the creation of two or four in-particles from the in-vacuum. (The construction of the in-vacuum and in-particle states is discussed in I.) The second part of the discussion, presented in § 3, deals with the renormalisation of vacuum-to-vacuum processes, that is, with $\langle \text{in}|S|\text{in}\rangle$ where $|\text{in}\rangle$ is the in-vacuum and S is

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the S-matrix operator. We find that the normal ordering we have performed removes the divergences from all closed loops which begin and end at the same point (in other words, from $G(x) \equiv G(x, x)$ where $G(x, x') = \langle \text{in}|T(\phi(x)\phi(x'))|\text{in}\rangle$) but is not sufficient to make all vacuum-to-vacuum processes finite. There is one diagram (see figure 1) which is unaffected by normal ordering and which in Minkowski space can be cancelled by adding an infinite constant to the Lagrangian or Hamiltonian density. In curved space–time it is not possible to remove divergences in this way since the energy of the vacuum has physical significance, being coupled to the geometry of space–time through Einstein's equation. The correct procedure for removing these divergences is by renormalising coupling constants in Einstein's equation or, equivalently, in the Einstein action. In § 3 we show that this is conveniently done by recognising that the sum of all vacuum-to-vacuum diagrams is related to the effective action for the field theory.

One of the principal calculations of the paper, the evaluation of the divergences in $G^3(x, x')$ which arises in figure 7 in I, appears in an appendix at the end of the paper. This calculation relies on a representation of the Feynman propagator as an integral over an n-dimensional momentum space. An outline of the derivation of this representation in a conformally flat space–time, which is sufficiently general to enable us to obtain the structure of the divergences in $G^3(x, x')$, will be given in the remainder of this section. Further details appear in Bunch and Parker (1980) where a momentum space representation valid in arbitrary space–times is derived and its relationship to the proper time representation (DeWitt 1975) and to the field theory formalism based on the concept of adiabatic particle states (Parker and Fulling 1974) is discussed. Consider an n-dimensional conformally flat space–time with metric:

$$g_{\mu\nu} = C(x)\eta_{\mu\nu} \tag{1.1}$$

where $\eta_{\mu\nu}$ is the Minkowski metric. The Feynman Green's function $G(x, x')$ for a free scalar field satisfies the wave equation:

$$\Box G(x, x') + (m^2 + \xi R)G(x, x') = -\mathrm{i}g^{-1/2}(x)\delta(x, x') \tag{1.2}$$

where

$$g = \det(-g_{\mu\nu}) = C^n(x). \tag{1.3}$$

Define

$$\bar{G}(x, x') = C^{(n-2)/4}(x)C^{(n-2)/4}(x')G(x, x'). \tag{1.4}$$

Then, using the expression for the Ricci scalar in a conformally flat space–time:

$$R = \frac{4(n-1)}{2-n} C^{(n-2)/4} \Box C^{(2-n)/4} \tag{1.5}$$

we find that $\bar{G}(x, x')$ satisfies the equation:

$$\eta^{\mu\nu}\partial_\mu\partial_\nu \bar{G}(x, x') + C[m^2 + (\xi - \xi(n))R]\bar{G}(x, x') = -\mathrm{i}\delta(x, x') \tag{1.6}$$

where

$$\xi(n) = \frac{n-2}{4(n-1)} = \frac{1}{6} + \frac{n-4}{12(n-1)}. \tag{1.7}$$

Equation (1.6) is satisfied by $\bar{G}(x, x')$ in either of the variables x or x'. It will be convenient for us to consider $\bar{G}(x, x')$ as a function of x' for each fixed point x, the reason being that in § 2 and in the appendix in which the divergences of $G^3(x, x')$ are evaluated, $G^3(x, x')$ is considered to be a distribution in x' for each fixed x. The divergences in $G^3(x, x')$ arise when an integration over x' is performed in a neighbourhood of $(x'-x)^2=0$, i.e. in a neighbourhood of the light cone of the fixed point x. If an analytic continuation of the metric to a metric with (negative definite) Euclidean signature is carried out the singularities appear when integrating in a neighbourhood of the point $x'=x$. Although we will not explicitly perform such an analytic continuation, we mention it since it could be used to provide a more rigorous mathematical justification of our arguments. In particular we are going to look for a solution of (1.6) valid in some neighbourhood of $(x'-x)^2=0$ and it will be convenient to think of this as being a neighbourhood of a point.

First expand the functions C and R in (1.6) in Taylor series about the point $x'=x$. It turns out that only the first term in each expansion affects the divergent part of $\bar{G}(x, x')$ so that we may consider the equation:

$$\eta^{\mu\nu}\partial_{\mu'}\partial_{\nu'}\bar{G}(x, x')+C(x)[m^2+(\xi-\xi(n))R(x)]\bar{G}(x, x')=-\mathrm{i}\delta(x, x'). \tag{1.8}$$

Express $\bar{G}(x, x')$ as a Fourier transform according to

$$\bar{G}(x, x')=\mathrm{i}\int\frac{\mathrm{d}^n k}{(2\pi)^n}\bar{G}(k)\exp[\mathrm{i}k(x'-x)] \tag{1.9}$$

with inverse

$$\bar{G}(k)=-\mathrm{i}\int\bar{G}(x, x')\exp[-\mathrm{i}k(x'-x)]\mathrm{d}^n x' \tag{1.10}$$

where it is to be understood that the scalar product $k(x'-x)$ is defined in n-dimensional Minkowski space. Thus

$$k(x'-x)\equiv\eta_{\mu\nu}k^\mu(x'-x)^\nu. \tag{1.11}$$

Equations (1.8) and (1.9) lead to an equation for the Fourier transform $\bar{G}(k)$:

$$[k^2-M^2-(\xi-\xi(n))CR]\bar{G}(k)=1 \tag{1.12}$$

where

$$C=C(x) \qquad R=R(x) \qquad M^2\equiv C(x)m^2 \qquad k^2=\eta_{\mu\nu}k^\mu k^\nu. \tag{1.13}$$

Thus we obtain

$$\bar{G}(k)\approx\frac{1}{k^2-M^2}+\frac{(\xi-\xi(n))CR}{(k^2-M^2)^2}+\mathrm{O}(k^{-6}). \tag{1.14}$$

Notice that the order k^{-4} term comes from taking $C(x')\approx C(x)$ and $R(x')\approx R(x)$ in (1.6). Higher order terms in the expansion of $C(x')$ and $R(x')$ give contributions to (1.14) which are of order k^{-6} or higher. These terms do not affect the divergences in $G(x, x')$ and so need not be calculated explicitly. Following I we define $G_\mathrm{R}(x)$ by

$$G(x)\equiv\lim_{x'\to x}G(x, x')=\frac{m^2+(\xi-\frac{1}{6})R}{8\pi^2(n-4)}+G_\mathrm{R}(x). \tag{1.15}$$

Then we find:

$$G(x, x')=C^{(2-n)/4}(x)C^{(2-n)/4}(x')\bigg[\mathrm{i}\int\frac{\mathrm{d}^n k}{(2\pi)^n}\exp[\mathrm{i}k(x'-x)]\bigg(\frac{1}{k^2-M^2}+\frac{\mathrm{i}(\xi-\xi(n))CR}{(k^2-M^2)^2}\bigg)$$
$$+C^{(n-2)/2}(x)\bigg(\frac{R}{288\pi^2}-\frac{m^2(\gamma-1)+(\xi-\frac{1}{6})R\gamma}{16\pi^2}\bigg)\bigg]+G_\mathrm{R}(x, x') \tag{1.16}$$

where

$$\lim_{x'\to x} G_R(x, x') = G_R(x). \tag{1.17}$$

The terms in (1.16) involving Euler's constant, γ, are required to cancel similar terms which arise when the n-dimensional momentum integrals are performed in the limit $x'\to x$.

2. Four- and two-point functions

In this section we study the divergences that arise in second-order matrix elements that lead to the creation of two or four particles from the vacuum. Various disconnected and one-particle reducible diagrams are obtained which are products of first-order contributions and will not be considered. They do, however, give finite, non-zero contributions that must be included if one is interested in calculating the amplitude for these processes.

The normal ordered interaction Hamiltonian density, including renormalisation counterterms is, to second order in λ,

$$\begin{aligned}\mathcal{H}'(x) = {}&\tfrac{1}{4}\lambda\sqrt{g(x)}[\phi_0^4(x) - 6\phi_0^2(x)G_D(x) + 3G_D^2(x)]\\ &+\tfrac{1}{2}\lambda^2\sqrt{g(x)}[Z_2^{(2)}m^2 + Z_3^{(2)}\xi R][\phi_0^2(x) - G_D(x)]\\ &+\tfrac{1}{4}Z_4^{(1)}\lambda^2\sqrt{g(x)}[\phi_0^4(x) - 6\phi_0^2(x)G_D(x) + 3G_D^2(x)]\end{aligned} \tag{2.1}$$

where we have used the same notation as in I. $G_D(x)$ is the divergent part of the coincidence limit of $G(x, x')$ and is given by (in n dimensions)

$$G_D(x) = \frac{m^2 + (\xi - \frac{1}{6})R}{8\pi^2(n-4)}. \tag{2.2}$$

We also write $G(x)$ for $G(x, x')|_{x'=x}$ and $G_R(x)$ for $G(x) - G_D(x)$ (see (1.15)). Second-order matrix elements between states in the 'in' Fock space are evaluated by using Wick's theorem on time-ordered products of field operators.

Performing this reduction and retaining only one-particle irreducible (1PI) diagrams gives the following contributions to the four-particle amplitude:

$$\frac{-\lambda^2}{32}\int \sqrt{g(x)g(x')}\, d^4x\, d^4x'\{72G^2(x, x')\}\phi_{p_1}(x)\ldots\phi_{p_4}(x') \tag{2.3a}$$

and

$$-\frac{i\lambda^2}{4}Z_4^{(1)}\int \sqrt{g(x)}\, d^4x\phi_{p_1}(x)\ldots\phi_{p_4}(x) \tag{2.3b}$$

where $\phi_{p_1}(x)$, etc, represent external wavefunctions. The factor of 72 arises from performing the Wick reduction. These two terms correspond to figures 4 and 5 in I. We have excluded from (2.3) a factor 4! which comes from permuting the external lines.

Figure 4 in I represents a divergence that must be removed by the coupling constant renormalisation shown in figure 5 in I. In I we discussed this divergence briefly. Here we present a slightly different derivation. As before we use dimensional regularisation to exhibit the divergences in terms of quantities that have poles at $n = 4$, where n is the dimensional parameter.

In Birrell and Taylor (1979) it was argued that the divergence in $G^2(x, x')$ can be represented as a δ function times an infinite number, reflecting the fact that the divergence arises owing to the coincidence of x and x' when the integrations are performed. We may thus write

$$G^2(x, x') = cg^{-1/2}(x)\delta(x, x') + \text{finite term} \tag{2.4}$$

where c is the infinite number. We know that $G(x, x')$ satisfies the inhomogeneous wave equation

$$(\Box + m^2 + \xi R)G(x, x') = -\mathrm{i}\tilde{\delta}(x, x'). \tag{2.5}$$

In this equation we have:

$$\tilde{\delta}(x, x') = g^{-1/2}(x)\delta(x, x') \tag{2.6}$$

to represent the covariant delta function. Differentiating with respect to m^2:

$$(\Box + m^2 + \xi R)\frac{\partial G}{\partial m^2}(x, x') = -G(x, x'). \tag{2.7}$$

Since this is a differential equation for $\partial G/\partial m^2$ and since $G(x, x')$ is the Green function for the wave operator we can immediately write the solution for $\partial G/\partial m^2$ as

$$\frac{\partial G}{\partial m^2}(x, x') = -\mathrm{i}\int G(x, x'')G(x'', x')\sqrt{g(x'')}\,\mathrm{d}^n x''. \tag{2.8}$$

Since we only want the divergent part of $G^2(x, x'')$ we may replace $\partial G/\partial m^2$ by the divergent part of its coincidence limit and use (2.4) in the right-hand side of (2.8), giving

$$\frac{\partial G_{\mathrm{D}}(x)}{\partial m^2} = -\mathrm{i}\int c\tilde{\delta}(x, x'')\sqrt{g(x'')}\,\mathrm{d}^n x'' = -\mathrm{i}c. \tag{2.9}$$

Using (2.2) we obtain

$$c = \frac{\mathrm{i}}{8\pi^2(n-4)}. \tag{2.10}$$

The divergence in (2.2) is thus

$$-\frac{9\mathrm{i}\lambda^2}{32\pi^2(n-4)}\int \sqrt{g(x)}\,\mathrm{d}^n x\,\phi_{p_1}(x)\ldots\phi_{p_4}(x) \tag{2.11}$$

and must be cancelled by the counterterm

$$-\frac{\mathrm{i}\lambda^2}{4}Z_4^{(1)}\int \sqrt{g(x)}\,\mathrm{d}^n x\,\phi_{p_1}(x)\ldots\phi_{p_4}(x). \tag{2.12}$$

From (2.11) and (2.12) we obtain the result

$$Z_4^{(1)} = -\frac{9}{8\pi^2(n-4)}. \tag{2.13}$$

This agrees with calculations performed in flat space–time (Collins 1974) and de Sitter space–time (Drummond 1975). Our result shows that coupling constant renormalisation is unaffected by curvature even though we have a field theory that is not conformally invariant and in which there is particle creation by the gravitational field.

We turn now to the two-point function. When the Wick reduction is performed one obtains various terms represented by figures 6–9 of I. As before, we consider only 1PI diagrams. We see that the normal ordering renders the self-closing loops finite and replaces them by $G_R(x)$. We note again that if one is interested in the actual amplitude to create two in-particles one must add the finite contributions from the disconnected and the reducible diagrams. The divergences that remain in the above diagrams must be cancelled by renormalising the physical parameters in the theory.

We have then the following integrals representing the Feynman diagrams in figures 6–9 of I respectively:

$$-\frac{9\lambda^2}{2}\int \sqrt{g(x)g(x')}\, d^nx\, d^nx' \phi_{p_1}(x')\phi_{p_2}(x')G_R(x)G^2(x, x') \tag{2.14}$$

$$-3\lambda^2 \int \sqrt{g(x)g(x')}\, d^nx\, d^nx' \phi_{p_1}(x)\phi_{p_2}(x')G^3(x, x') \tag{2.15}$$

$$-\frac{3i\lambda^2}{2} Z_4^{(1)} \int \sqrt{g(x)}\, d^nx G_R(x)\phi_{p_1}(x)\phi_{p_2}(x) \tag{2.16}$$

$$-\frac{i\lambda^2}{2}\int \sqrt{g(x)}\, d^nx \phi_{p_1}(x)\phi_{p_2}(x)[m^2 Z_2^{(2)} + \xi R Z_3^{(2)}]. \tag{2.17}$$

(2.14) is still divergent because it contains $G^2(x, x')$ and (2.16) is divergent because it contains the infinite (at $n=4$) number $Z_4^{(1)}$. We can evaluate the divergence in (2.14) using (2.4) and (2.10) and that in (2.16) is given by (2.13). Combining (2.14) and (2.16) we finally obtain

$$\frac{9i\lambda^2}{8\pi^2(n-4)}\int \sqrt{g(x)}\, d^nx \phi_{p_1}(x)\phi_{p_2}(x)G_R(x). \tag{2.18}$$

We thus have a divergence depending on $G_R(x)$. This is a quantity that depends on the choice of the in-vacuum and cannot be cancelled by any counterterm. In order that the theory be renormalisable it must cancel a contribution from (2.15). It is to check this cancellation that one requires the evaluation of $Z_4^{(1)}$ and $G^3(x, y)$.

Birrell and Taylor (1979) have shown how one can isolate the divergences in $G^3(x, x')$ by expressing it as a distribution involving the delta function and its derivatives. In I we discussed this issue and concluded that $G^3(x, x')$ could contain a term proportional to $G_R(x)$. In the appendix the evaluation of $G^3(x, x')$ is given in detail. Here we note that the calculation yields

$$G^3(x, x') = \left[\left(\frac{3i}{128\pi^4(n-4)^2} - \frac{3i}{256\pi^4(n-4)}\right)(m^2 + (\xi - \tfrac{1}{6})R) + \frac{iR}{3072\pi^4(n-4)} + \frac{3iG_R(x)}{8\pi^2(n-4)}\right]\tilde{\delta}(x, x') - \frac{i}{512\pi^4(n-4)}\Box'\tilde{\delta}(x, x') + \text{finite term}. \tag{2.19}$$

The coefficient of the $\Box'\tilde{\delta}(x, x')$ term gives the wavefunction renormalisation. This can be seen by noting that in flat space–time one can Fourier transform (2.19). The $\Box'\tilde{\delta}(x, x')$ would then become a term in p^2 (where p is the four-momentum) in the proper self-energy. This would require a rescaling of the propagator if one imposes the condition that the full propagator have a pole at $p^2 = m^2$ with residue one, the usual condition for fixing wavefunction renormalisation.

If we insert (2.19) in (2.15) and use the δ function to perform one of the integrals we obtain terms involving m^2, R and $G_R(x)$. The term involving $G_R(x)$ is

$$-3\lambda^2 \int \sqrt{g(x)}\, d^n x \phi_{p_1}(x)\phi_{p_2}(x) \frac{3i}{8\pi^2(n-4)} G_R(x) \tag{2.20}$$

which precisely cancels (2.18). We have thus shown that the state-dependent divergence has cancelled. The remaining terms result in the renormalisation of m^2 and ξ.

The term involving $\Box'\tilde{\delta}(x, x')$ becomes, when integrated,

$$\frac{3i\lambda^2}{512\pi^4(n-4)} \int \sqrt{g(x)}\, d^n x \phi_{p_1}(x)\Box\phi_{p_2}(x). \tag{2.21}$$

Since $\phi_{p_2}(x)$ is a solution of the free wave equation, we have

$$\Box\phi_{p_2}(x) = -(m^2 + \xi R)\phi_{p_2}(x). \tag{2.22}$$

Thus the divergence in (2.15) proportional to m^2 is

$$-\left(\frac{9i\lambda^2}{128\pi^4(n-4)^2} - \frac{15i\lambda^2}{512\pi^4(n-4)}\right) m^2 \int \sqrt{g(x)}\, d^n x \phi_{p_1}(x)\phi_{p_2}(x). \tag{2.23}$$

Comparing this with (2.17) gives us the result

$$Z_2^{(2)} = -\frac{9}{64\pi^4(n-4)^2} + \frac{15}{256\pi^4(n-4)}. \tag{2.24}$$

The calculation of $Z_3^{(2)}$ is virtually identical and gives a value of $Z_3^{(2)}$ that is most compactly written as

$$\xi Z_3^{(2)} = (\xi - \tfrac{1}{6}) Z_2^{(2)} - \frac{1}{256\pi^4(n-4)}. \tag{2.25}$$

In our treatment of the two-point function we considered the special case where two particles are created from the vacuum. In this situation wavefunction renormalisation turns out to be unnecessary. If, however, figure 7 in I is a subdiagram of a larger diagram then we obtain an additional divergence that has to be cancelled by rescaling the propagator. The full propagator $\tilde{G}(x, x')$ is given (to second order in λ) by

$$\tilde{G}(x, x') = Z_1^{-1} G(x, x') + Z_1^{-1} \sum_{n=1,2} \langle \mathrm{in}| T\phi(x)\phi(x') S^{(n)} |\mathrm{in}\rangle \tag{2.26}$$

where Z_1 is the wavefunction renormalisation constant. There is no wavefunction renormalisation in first order so in second order in λ the relevant terms are

$$\tilde{G}(x, x') = -\lambda^2 Z_1^{(2)} G(x, x') + \langle \mathrm{in}| T\phi(x)\phi(x') S^{(2)} |\mathrm{in}\rangle. \tag{2.27}$$

If we consider (2.15) with propagators instead of wavefunctions for the external lines we have

$$-6\lambda^2 \int \sqrt{g(y)g(y')}\, d^n y\, d^n y' G(x, y) G^3(y, y') G(y', x') \tag{2.28}$$

where an extra factor of two is introduced to take into account the possibility of permuting external legs. If we insert (2.19) in (2.28) and note that $G(y', x')$ satisfies the inhomogeneous wave equation we get a term $3\lambda^2 G(x, x')/[256\pi^4(n-4)]$ which can only be cancelled by rescaling the propagator. This rescaling is the same for any space–time and in the appendix we make use of the known rescaling in Minkowski space to fix the coefficient of $\Box'\tilde{\delta}(x, x')$ in (2.19).

We have shown in this section that the mass, wavefunction and coupling constant renormalisations are the same as in flat space–time (see, for example, Collins 1974). In addition we find that in curved space–time a renormalisation of the coupling to the scalar curvature is required. Drummond (1975) has also considered $\lambda\phi^4$ theory in de Sitter space–time. However, his theory was massless and conformally invariant and hence did not have any mass renormalisation or any state-dependent divergences in the intermediate steps.

The divergences we have considered so far seem to be the only basic ones in the theory. Divergent diagrams in higher order would only arise because they contain subdiagrams of the type we have discussed here. Furthermore, the topology of the diagrams is the same as in flat space. Thus we expect that a proof of the normalisability to all orders, by induction, would be a straightforward modification of the usual flat space proofs if one ignores vacuum polarisation diagrams.

3. Vacuum polarisation

In I we discussed the generalisation of normal ordering to curved space–time and showed how this rendered self-closing loops finite. In second-order processes there are vacuum diagrams that are not made finite by normal ordering (see figures 1 and 2).

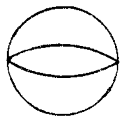


Figure 1.

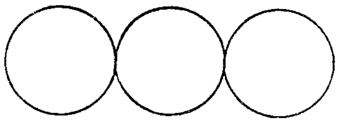


Figure 2.

Second-order diagrams that are disconnected are products of first-order diagrams that are rendered finite by normal ordering and we do not consider them further. There are also vacuum diagrams arising from the counterterms (figures 3 and 4).

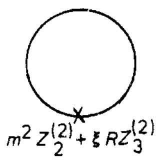
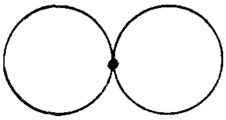


Figure 3.



$Z_4^{(1)}$

Figure 4.

We do not find that divergences in all vacuum diagrams cancel. Instead we shall show that all state-dependent divergences do cancel and then argue that the remaining divergences can be removed by renormalising coupling constants in the Einstein action.

Performing the normal ordering renders the self-closing loops finite and replaces the $G(x)$ factor with $G_R(x)$. The contribution of the diagrams in figures 1–4 is

$$\begin{aligned}\langle \text{in}|S^{(2)}|\text{in}\rangle = &-\frac{3\lambda^2}{4}\int G^4(x,x')\sqrt{g(x)g(x')}\,\mathrm{d}^n x\,\mathrm{d}^n x' \\ &-\frac{9}{4}\lambda^2\int G^2(x,x')G_R(x)G_R(x')\sqrt{g(x)g(x')}\,\mathrm{d}^n x\,\mathrm{d}^n x' \\ &-\frac{\mathrm{i}}{2}\lambda^2\int (Z_2^{(2)}m^2+Z_3^{(2)}\xi R)G_R(x)\sqrt{g(x)}\,\mathrm{d}^n x \\ &-\frac{3\mathrm{i}\lambda^2}{4}Z_4^{(1)}\int G_R^2(x)\sqrt{g(x)}\,\mathrm{d}^n x. \end{aligned} \tag{3.1}$$

We may write

$$G(x,x') = G_D(x,x') + G_R(x,x') \tag{3.2}$$

where $G_R(x, x')$ is defined by (1.16). The term in $G^4(x, x')$ is

$$G^4(x,x') = G_D^4(x,x') + 4G_D^3(x,x')G_R(x,x') + 6G_D^2(x,x')G_R^2(x,x') + \ldots. \tag{3.3}$$

The last two terms in $G^4(x, x')$ involve G_R^3 and G_R^4 and are finite. If we use (2.4), (2.10) and (2.13) in (3.1) we see that terms involving G_R^2 all cancel. We note that G_D^3 is the same as the divergent part of G^3 with the G_R term omitted. 'f we use (2.19) in (3.1) and rewrite the term in $\Box' G_R$ as follows:

$$\Box' G_R(x,x')|_{x'=x} = K' G_R(x,x')|_{x'=x} - [m^2+\xi R(x)]G_R(x) \tag{3.4}$$

where K' is the wave operator

$$K' \equiv \Box' + m^2 + \xi R \tag{3.5}$$

we see that all terms in $G_R(x)$ also cancel, leaving only a term in G_D^4 and one in KG_R.

The term in G_D^4 can only involve m^4, $m^2 R$ and R^2 since there is no state-dependent piece in G_D itself. We will show that KG_R is also independent of the choice of vacuum state.

We can write

$$K' G_R(x,x') = K'(G(x,x') - G_D(x,x')) \tag{3.6}$$

where the prime on the K signifies that it acts on x'. Thus

$$K' G_R(x,x') = -\mathrm{i}g^{-1/2}(x)\delta(x,x') - K' G_D(x,x'). \tag{3.7}$$

Using (1.4) we have

$$G_D(x, x') = C^{(2-n)/4}(x)C^{(2-n)/4}(x')\bar{G}_D(x, x') \tag{3.8}$$

and hence

$$K'G_D(x, x') = C^{(2-n)/4}(x)C^{-(n+2)/4}(x')\{\eta^{\mu'\nu'}\partial_{\mu'}\partial_{\nu'}\bar{G}_D(x, x') + C[m^2 + (\xi - \xi(n))R]\bar{G}_D(x, x')\}. \tag{3.9}$$

Using the momentum space representation for $\bar{G}_D$ (see (1.16))

$$\bar{G}_D(x, x') = \mathrm{i}\int \frac{\mathrm{d}^n k}{(2\pi)^n} \exp[\mathrm{i}k(x'-x)]\left(\frac{1}{k^2 - M^2} + \frac{\mathrm{i}(\xi - \xi(n))CR}{(k^2 - M^2)^2}\right) + C^{(n-2)/2}(x)\left(\frac{R(x)}{288\pi^2} - \frac{m^2(\gamma - 1) + (\xi - \frac{1}{6})R(x)\gamma}{16\pi^2}\right) \tag{3.10}$$

in equation (3.9) we obtain

$$K'G_D(x, x') = C^{(2-n)/4}(x)C^{-(n+2)/4}(x') \times \left(-\mathrm{i}\delta(x, x') - \mathrm{i}(\xi - \xi(n))CR\int \frac{\mathrm{d}^n k}{(2\pi)^n}\frac{\exp[\mathrm{i}k(x'-x)]}{(k^2 - M^2)^2} + C[m^2 + (\xi - \xi(n))R]\bar{G}_D(x, x')\right). \tag{3.11}$$

It now follows from (3.7), (3.10) and (3.11) that $K'G_R(x, x')|_{x'=x}$ involves only m^2 and the Ricci scalar R.

Drummond and Shore (1979) have discussed figure 1 for a massless theory in de Sitter space–time. They find that it is finite at $n = 4$ but in our theory (which is not conformally invariant) there are divergences not removable by renormalisation or normal ordering.

Now that we have shown that the divergences which remain in second-order vacuum-to-vacuum processes depend only on the mass of the field and the Ricci scalar, R, we will outline how these divergences are to be removed. The field theory based on the Lagrangian density given by equation (2.1) of I is incomplete because we have not yet indicated how the quantum matter is to be coupled to the geometry of space–time. In Schwinger's formulation of field theory this is achieved by coupling the effective action, W, of the field theory to the gravitational action, S_G. The effective action is related to the vacuum persistence amplitude by

$$\mathrm{e}^{\mathrm{i}W} = \langle \mathrm{out}|S|\mathrm{in}\rangle. \tag{3.12}$$

Inserting a complete set of in-states,

$$\mathrm{e}^{\mathrm{i}W} = \langle \mathrm{out}|\mathrm{in}\rangle\langle \mathrm{in}|S|\mathrm{in}\rangle + \int \mathrm{d}\mu(\boldsymbol{p}_1)\,\mathrm{d}\mu(\boldsymbol{p}_2)\langle \mathrm{out}|\boldsymbol{p}_1, \boldsymbol{p}_2, \mathrm{in}\rangle\langle \mathrm{in}, \boldsymbol{p}_1, \boldsymbol{p}_2|S|\mathrm{in}\rangle + \ldots. \tag{3.13}$$

Every physical process occurring in $\langle \mathrm{in}|S|\mathrm{in}\rangle$, $\langle \mathrm{in}, \boldsymbol{p}_1, \boldsymbol{p}_2|S|\mathrm{in}\rangle$, etc, is modified by a factor which may be expressed as e^{Γ} where Γ is the mathematical equivalent of figure 1. (This argument for factoring out vacuum-to-vacuum processes can be found in many textbooks on quantum field theory; see, for example, Schweber (1961, p 470).) Thus we may rewrite (3.12) as:

$$\mathrm{e}^{\mathrm{i}W} = \mathrm{e}^{\Gamma}\langle \mathrm{out}|\tilde{S}|\mathrm{in}\rangle \tag{3.14}$$

where $\tilde{S}$ is obtained by modifying the S matrix so that it no longer gives rise to any diagrams containing figure 1. We have shown that once the state-dependent divergences in Γ are cancelled with similar divergences in other vacuum-to-vacuum diagrams, the remaining divergences, Γ_D, are expressible in terms of the constant m and the Ricci scalar R. Thus we may write

$$e^{iW-\Gamma_D} = e^{\Gamma_R}\langle\text{out}|\tilde{S}|\text{in}\rangle \tag{3.15}$$

where Γ_R is the finite remainder from figure 1, and $\langle\text{out}|\tilde{S}|\text{in}\rangle$ has been made finite, at least to second order, by our earlier work. The coupled theory is now unchanged if we replace $W+S_G$ by $\tilde{W}+\tilde{S}_G$ where

$$\tilde{W} = W + i\Gamma_D \tag{3.16}$$

$$\tilde{S}_G = S_G - i\Gamma_D. \tag{3.17}$$

But

$$S_G = -(8\pi G)^{-1}\int [\Lambda - \tfrac{1}{2}R + \tfrac{1}{2}\lambda_1 R^2 + \tfrac{1}{2}\lambda_2 R^{\alpha\beta}R_{\alpha\beta} + \tfrac{1}{2}\lambda_3 R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}]\sqrt{g(x)}\,d^n x \tag{3.18}$$

where terms quadratic in the curvature are included to absorb divergences. If desired, their renormalised coupling constants can be set to zero. Moreover, we have shown that

$$\Gamma_D = i\int (\alpha_1 m^4 + \alpha_2 m^2 R + \alpha_3 R^2)\sqrt{g(x)}\,d^n x \tag{3.19}$$

where α_1, α_2, α_3 are, as can be readily verified, real quantities depending only on n and having poles at $n=4$. Thus (3.17) corresponds to a renormalisation of the coupling constants in a modified form of Einstein's equation.

In paper I we introduced a generalisation of normal ordering to curved space–time and we have continued to make use of this procedure in the present paper. However, we have found that normal ordering is not sufficient to remove all divergences from vacuum-to-vacuum diagrams and renormalisation of coupling constants is required to remove the remaining divergences. In retrospect, it is clear that this renormalisation of the gravitational action could have been used to renormalise vacuum-to-vacuum processes even if normal ordering had not been performed. However, as we found in paper I normal ordering is sometimes a very convenient procedure to use and its physical justification is given by this equivalence to the renormalisation of gravitational coupling constants.

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Since completing this work we have received notification from N D Birrell that he has obtained similar results for the renormalisation of two- and four-point Green's functions up to second order in λ. Birrell also uses a momentum space technique but his approach is somewhat different from ours. We would like to thank him for pointing out an error in an early version of our work.

Appendix. Evaluation of divergences in $G^3(x, x')$

It was shown in § 5 of I, on the basis of arguments of Birrell and Taylor (1979), that the structure of $G^3(x, x')$ as a distribution in x', is

$$G^3(x, x') = [c_0 m^2 + c_1 R + c_2 G_R(x)]\tilde{\delta}(x, x') + c_3 \Box' \tilde{\delta}(x, x') + \text{finite term} \tag{A1}$$

where c_0, c_1, c_2 and c_3 are functions only of n which have poles at $n = 4$, $\tilde{\delta}(x, x')$ is the invariant delta function $g^{-1/2}(x')\delta(x, x')$, and the finite term is a distribution which maps test functions in n dimensions to quantities which are finite when $n = 4$. In § 2 of this paper, it was shown that the quantity c_3 determines the wavefunction renormalisation which is to be carried out. This renormalisation is the same for all space–times and so c_3 can be determined from wavefunction renormalisation in Minkowski space–time (Collins 1974). We find

$$c_3 = -\frac{\mathrm{i}}{512\pi^4(n-4)}. \tag{A2}$$

The coefficient of $\tilde{\delta}(x, x')$ in (A1) is given by

$$\int G^3(x, x')\sqrt{g(x')}\, \mathrm{d}^n x'. \tag{A3}$$

We will evaluate (A3) and thus obtain the coefficients c_0, c_1 and c_2. Notice that this approach will also serve to verify that the coefficient of $\tilde{\delta}(x, x')$ on (A1) is indeed a linear combination of m^2, R and $G_R(x)$ (at least in conformally flat space–times).

First express (A3) in terms of $\bar{G}(x, x')$ which is given by (1.4) and (1.16):

$$\int G^3(x, x')C^{n/2}(x')\, \mathrm{d}^n x' = C^{3(2-n)/4}(x) \int \bar{G}^3(x, x')C^{(6-n)/4}(x')\, \mathrm{d}^n x'. \tag{A4}$$

Now expand $C^{(6-n)/4}(x')$ about $x' = x$; then (A4) becomes

$$\begin{aligned}\int G^3(x, x')C^{n/2}(x')\, \mathrm{d}^n x' \\ \approx C^{(3-n)}(x) \int \bar{G}^3(x, x')\, \mathrm{d}^n x' + \tfrac{1}{8}(6-n)C^{3-n}(x) \\ \times [C^{-1}\partial_\mu\partial_\nu C + \tfrac{1}{4}(2-n)C^{-2}\partial_\mu C\partial_\nu C] \int (x'-x)^\mu (x'-x)^\nu \bar{G}^3(x, x')\, \mathrm{d}^n x'.\end{aligned} \tag{A5}$$

But

$$\bar{G}(x, x') = I_1(x, x') + I_2(x, x') + F(x, x') \tag{A6}$$

where

$$I_1(x, x') = \mathrm{i} \int \frac{\mathrm{d}^n k}{(2\pi)^n} \frac{\exp[\mathrm{i}k(x'-x)]}{k^2 - M^2} \tag{A7}$$

$$I_2(x, x') = (\xi - \xi(n))CR\frac{\partial I_1}{\partial M^2}(x, x') \tag{A8}$$

$$\begin{aligned}F(x, x') = C^{(n-2)/2}(x)\left(\frac{R}{288\pi^2} - \frac{m^2(\gamma-1) + (\xi - \frac{1}{6})R\gamma}{16\pi^2}\right) \\ + C^{(n-2)/4}(x)C^{(n-2)/4}(x')G_R(x, x').\end{aligned} \tag{A9}$$

Therefore

$$\int \bar{G}^3(x, x')\,d^nx'$$

$$= \int I_1^3(x, x')\,d^nx' + (\xi - \xi(n))R\frac{\partial}{\partial m^2}\int I_1^3(x, x')\,d^nx'$$

$$+ 3\int I_1^2(x, x')F(x, x')\,d^nx' + \text{finite terms.} \tag{A10}$$

The first integral can be determined from Minkowski space–time (see, for example, Collins 1974). The result obtained is

$$\int I_1^3(x, x')\,d^nx' \approx \frac{iM^{2(n-3)}}{256\pi^4}\left(\frac{6}{(n-4)^2} - \frac{3}{(n-4)} + \frac{6(\gamma-1)}{n-4}\right). \tag{A11}$$

Moreover, we know that

$$I_1^2(x, x') \approx \frac{i}{8\pi^2(n-4)}C^{(n-4)/2}(x')\delta(x, x') \tag{A12}$$

since this is the divergent part of the distribution $\bar{G}^2(x, x')$. Hence, from (A9)–(A12), we obtain:

$$\int \bar{G}^3(x, x')\,d^nx'$$

$$= C^{n-3}(x)\left[\left(\frac{3i}{128\pi^4(n-4)^2} - \frac{3i}{256\pi^4(n-4)}\right)[m^2 + (\xi - \tfrac{1}{6})R]\right.$$

$$\left. + \frac{iR(x)}{1536\pi^4(n-4)} + \frac{3iG_R(x)}{8\pi^2(n-4)}\right]. \tag{A13}$$

Finally, consider the contribution from the second term in (A5). We can write

$$\bar{G}^3(x, x') \equiv C^{3(n-2)/4}(x)C^{3(n-2)/4}(x')G^3(x, x')$$

$$= [c_0m^2 + c_1R + c_2G_R]C^{n-3}(x')\delta(x, x')$$

$$+ c_3C^{3(n-2)/4}(x)C^{3(n-2)/4}(x')\Box'(C^{-n/2}(x')\delta(x, x')). \tag{A14}$$

The second term in (A5) is then

$$\tfrac{1}{8}(6-n)C^{(6-n)/4}(x)[C^{-1}\partial_\mu\partial_\nu C + \tfrac{1}{4}(2-n)C^{-2}\partial_\mu C\partial_\nu C]$$

$$\times c_3\int (x'-x)^\mu(x'-x)^\nu C^{3(n-2)/4}(x')\Box'(C^{-n/2}\delta(x, x'))\,d^nx'. \tag{A15}$$

The integral in (A15) is simply

$$\int g^{\alpha\beta}\nabla'_\alpha\nabla'_\beta[(x'-x)^\mu(x'-x)^\nu C^{3(n-2)/4}(x')]C^{-n/2}(x')\delta(x, x')\,d^nx' = 2g^{\mu\nu}C^{(n-6)/4}(x). \tag{A16}$$

Thus (A15) reduces to

$$\tfrac{1}{4}c_3(6-n)\eta^{\mu\nu}[C^{-2}\partial_\mu\partial_\nu C + \tfrac{1}{4}(2-n)C^{-3}\partial_\mu C\partial_\nu C] \tag{A17}$$

but

$$R = (n-1)\eta^{\mu\nu}[C^{-2}\partial_\mu\partial_\nu C + \tfrac{1}{4}(n-6)C^{-3}\partial_\mu C\partial_\nu C]. \tag{A18}$$

Thus (A17) may be written as

$$\tfrac{1}{4}c_3\frac{6-n}{n-1}[R-\tfrac{1}{2}(n-4)C^{-3}\partial_\mu C\partial^\mu C]. \tag{A19}$$

Using (A2), the divergence in (A19) is just

$$-\frac{\mathrm{i}R}{3072\pi^4(n-4)}. \tag{A20}$$

Therefore, the constants c_0, c_1 and c_2 are:

$$c_0=\frac{3\mathrm{i}}{128\pi^4(n-4)^2}-\frac{3\mathrm{i}}{256\pi^4(n-4)} \tag{A21}$$

$$c_1=(\xi-\tfrac{1}{6})c_0+\frac{\mathrm{i}}{3072\pi^4(n-4)} \tag{A22}$$

$$c_2=\frac{3\mathrm{i}}{8\pi^2(n-4)}. \tag{A23}$$

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